GENERALIZED IMPULSE RESPONSES

ANDERS WARNE

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ABSTRACT: This note discusses how to compute generalized impulse responses and their asymptotic distribution. The results I present are essentially vector versions of what has already been shown by, e.g., Pesaran and Shin (1998). The value added is therefore measurable in terms of providing simpler algorithms for writing the computer code needed to make use of generalized impulse responses in practise.

KEYWORDS: Asymptotics, Impulse Response Function.

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1. SETUP

In contrast with impulse response functions for structural models, generalized impulse responses do not require that we identify any structural shocks. Accordingly, generalized impulse responses cannot explain how, say, inflation reacts to a monetary policy shock. Instead, generalized impulse responses provides a tool for describing the dynamics in a time series model by mapping out the reaction in, say, inflation to a one standard deviation shock to the residual in the interest rate equation.

The general setup we shall consider is a VAR process for some p dimensional time series x_t given by

$$x_t = \Phi D_t + \sum_{i=1}^{k} \Pi_i x_{t-i} + \varepsilon_t, \quad t = 1, \dots, T, \tag{1}$$

where D_t is a vector with deterministic variables. The process x_t may be covariance stationary, integrated of order d (and possibly cointegrated), while ε_t is p dimensional and assumed to be i.i.d. with zero mean and positive definite covariance matrix Ω.

The h-steps ahead forecast error for x_t is given by:

$$x_{t+h} - E\left[x_{t+h} | \mathbb{I}_t\right] = \sum_{j=0}^{h-1} C_j \varepsilon_{t+h-j}, \tag{2}$$

where $\mathbb{I}_t$ is an information set which includes the history of x_s up to and including period t as well as the entire time path for D_t. The $p \times p$ matrices C_j are given by $C_0 = I_p$ and

$$C_j = \sum_{i=1}^{\min k,j} \Pi_i C_{j-i}, \quad j \geq 1,$$

so that all C_j matrices can be determined recursively from the Π_i matrices.

Koop, Pesaran and Potter (1996) defined the generalized impulse response function by:

$$GI_x(h, \delta, \mathbb{I}_{t-1}) = E\left[x_{t+h} | \varepsilon_t = \delta, \mathbb{I}_{t-1}\right] - E\left[x_{t+h} | \mathbb{I}_{t-1}\right], \tag{3}$$

where δ is some known vector. For the VAR process this means that:

$$GI_x(h, \delta, \mathbb{I}_{t-1}) = C_h \delta.$$

The choice of δ is therefore central to determining the time profile for any generalized impulse response function. As an alternative to shocking all elements of ε_t one may consider just shocking one element such that $\varepsilon_{jt} = \delta_j$. We may now define the generalized impulse responses as:

$$GI_x(h, \delta_j, \mathbb{I}_{t-1}) = E\left[x_{t+h}|\varepsilon_{jt} = \delta_j, \mathbb{I}_{t-1}\right] - E\left[x_{t+h}|\mathbb{I}_{t-1}\right]. \tag{4}$$

Letting $\delta_j = \sqrt{\omega_{jj}}$, the standard deviation of ε_{jt}, and assuming that ε_t is Gaussian, it follows that:

$$E\left[\varepsilon_t|\varepsilon_{jt} = \sqrt{\omega_{jj}}\right] = \Omega e_j \omega_{jj}^{-1/2}, \tag{5}$$

where e_j is the j:th column of I_p. For the VAR model we then find that

$$GI_x(h, \sqrt{\omega_{jj}}, \mathbb{I}_{t-1}) = C_h \Omega e_j \omega_{jj}^{-1/2}.$$

This measures the response in x_{t+h} from a one standard deviation shock to ε_{jt}, where the correlation between ε_{jt} and ε_{it} is taken into account. Defining the diagonal $p \times p$ matrix Σ as:

$$\Sigma = \mathrm{diag}\begin{bmatrix} \left(e_1' \Omega e_1\right)^{-1/2} \\ \left(e_2' \Omega e_2\right)^{-1/2} \\ \vdots \\ \left(e_p' \Omega e_p\right)^{-1/2} \end{bmatrix}, \tag{6}$$

we may express the generalized impulse responses in matrix form as:

$$GI_x(h, \sqrt{\omega_{11}}, \dots, \sqrt{\omega_{pp}}, \mathbb{I}_{t-1}) = C_h \Omega \Sigma = C_h B = A_h, \tag{7}$$

where column j is given by $GI_x(h, \sqrt{\omega_{jj}}, \mathbb{I}_{t-1})$. When Ω is diagonal, then $B = \Omega^{1/2} = \Sigma^{-1}$, a diagonal matrix with standard deviations along the diagonal.

2. Asymptotics

In order to determine to asymptotic covariance matrix for an estimate of $C_h B$ we need to make a few assumptions. Suppose that C_h depends on a K dimensional vector $\theta \in \mathbb{R}^K$ and that C_h is differentiable with respect to θ. Relative to the VAR model, θ includes the elements of Π_i or some transformations thereof, but they do not include any element from Φ or Ω. In case the VAR model includes cointegration rank restrictions, then θ does not include the cointegration vectors but only the parameters on stationary transformations of x. For the VAR model where x_t is cointegrated of order (1,1), this means that θ only includes parameters on lagged first difference of x_t and on the $0 < r < p$ cointegration relations $\beta' x_{t-1}$.

Furthermore, assume that we have an estimator of θ, denoted by $\hat{\theta}$, based on a sample of T observations, which satisfies:

$$\sqrt{T}\left(\hat{\theta} - \theta\right) \xrightarrow{\mathrm{d}} N_K\left(0, \Sigma_\theta\right), \tag{8}$$

with N_K being a K-dimensional Gaussian distribution, $\xrightarrow{\mathrm{d}}$ denoting convergence in distribution, and Σ_θ being positive semidefinite. Furthermore, let $\omega = \mathrm{vech}(\Omega)$, with vech being the column stacking operator which only takes the elements on and below the diagonal. The estimator of ω, denoted by $\hat{\omega}$, is assumed to satisfy:

$$\sqrt{T}\left(\hat{\omega} - \omega\right) \xrightarrow{\mathrm{d}} N_{p(p+1)/2}\left(0, \Sigma_\omega\right), \tag{9}$$

while $\hat{\theta}$ and $\hat{\omega}$ are asymptotically independent. In case ε_t is Gaussian and, e.g., x_t is cointegrated of order (1,1) these assumptions are all satisfied as long as there are no restrictions which involve both θ and ω. Furthermore, for such a model

$$\Sigma_\omega = 2D_p^+\left(\Omega \otimes \Omega\right)D_p^{+'},$$

where $\otimes$ is the Kronecker product, D_p is the duplication matrix (cf. Magnus and Neudecker, 1988), and $D_p^+ = (D_p' D_p)^{-1} D_p'$ is the Moore-Penrose inverse of D_p.

Given our assumptions it follows that the asymptotic distribution of the matrix form of the generalized impulse responses in equation (7) can be expressed as:

$$\sqrt{T}\Big(\mathrm{vec}\big(\hat{A}_h\big) - \mathrm{vec}\big(A_h\big)\Big) \xrightarrow{\mathrm{d}} N_{p^2}(0, \Sigma_{A_h}), \tag{10}$$

where

$$\Sigma_{A_h} = \left[B' \otimes I_p\right] \frac{\partial \mathrm{vec}(C_h)}{\partial \theta'} \Sigma_\theta \left(\left[B' \otimes I_p\right] \frac{\partial \mathrm{vec}(C_h)}{\partial \theta'} \right)' + \left[I_p \otimes C_h\right] \frac{\partial \mathrm{vec}(B)}{\partial \omega'} \Sigma_\omega \left(\left[I_p \otimes C_h\right] \frac{\partial \mathrm{vec}(B)}{\partial \omega'} \right)'.$$

The partial derivatives $\partial \mathrm{vec}(C_h) / \partial \theta'$ are readily available from several sources (see, e.g., Warne, 1993, or Vlaar, 2004). Hence, what remains to be shown is what the matrix with partial derivatives $\partial \mathrm{vec}(B) / \partial \omega'$ looks like.

It can be shown that:

$$\frac{\partial \mathrm{vec}(\Sigma)}{\partial \mathrm{vec}(\Omega)'} = -\frac{1}{2} L_p \Sigma^3 L_p',$$

where L_p is a $p^2 \times p$ 0-1 matrix defined by

$$L_p = \begin{bmatrix} e_1 e_1' \\ e_2 e_2' \\ \vdots \\ e_p e_p' \end{bmatrix}.$$

It then follows that the differential of $\mathrm{vec}(B)$ satisfies:

$$d\mathrm{vec}(B) = \left[\Sigma \otimes I_p\right] d\mathrm{vec}(\Omega) - \frac{1}{2} \left[I_p \otimes \Omega\right] L_p \Sigma^3 L_p' d\mathrm{vec}(\Omega).$$

Since $d\mathrm{vec}(\Omega) = D_p d\omega$ we have that:

$$\frac{\partial \mathrm{vec}(B)}{\partial \omega'} = \left(\left[\Sigma \otimes I_p\right] - \frac{1}{2} \left[I_p \otimes \Omega\right] L_p \Sigma^3 L_p' \right) D_p. \tag{11}$$

If x_t is cointegrated of order (1,1) with r cointegration vectors, denoted by the full rank $p \times r$ matrix β, we may also define generalized impulse responses for the cointegration relations. Suppose that we have an estimator of β, denoted by $\hat{\beta}$, such that

$$\sqrt{T}\left(\hat{\beta} - \beta\right) \xrightarrow{\mathrm{p}} 0,$$

where $\xrightarrow{\mathrm{p}}$ denotes convergence in probability. Estimators of β, such as the ML estimator suggested by Johansen (1996), typically satisfy this assumption. Let the cointegrating relations be defined by $z_t = \beta' x_t$. The generalized impulse response function for z_{t+h} from one standard deviation shocks to ε_t is then given by

$$GI_z(h, \sqrt{\omega_{11}}, \ldots, \sqrt{\omega_{pp}}, \mathbb{I}_{t-1}) = \beta' A_h,$$

It can now be established that an estimator of $\beta' A_h$ satisfies:

$$\sqrt{T}\left(\mathrm{vec}\left(\hat{\beta}' \hat{A}_h\right) - \mathrm{vec}\left(\beta' A_h\right)\right) \xrightarrow{\mathrm{d}} N_{p^2}(0, [I_p \otimes \beta'] \Sigma_{A_h} [I_p \otimes \beta]). \tag{12}$$

The reason for this result is, of course, that $\hat{\beta}$ is $\sqrt{T}$-consistent whereas $\hat{A}_h$ is consistent.

3. Remarks

If x_t is cointegrated of order (1,1), we may rewrite the VAR in VEC form such that

$$\Delta x_t = \Phi D_t + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \alpha \beta' x_{t-1} + \varepsilon_t,$$

where α and β are full rank $p \times r$ matrices ($0 < r < p$); see, e.g., Johansen (1996) for details. In this case we may define $\theta = \mathrm{vec}([\Gamma_1 \ \cdots \ \Gamma_{k-1} \ \alpha])$ and the $p \times p$ matrix:

$$C = \beta_\perp \left(\alpha_\perp' \Gamma \beta_\perp\right)^{-1} \alpha_\perp',$$

with $\Gamma = I_p - \sum_{i=1}^{k-1} \Gamma_i$. We now find that

$$\lim_{h \to \infty} A_h = CB,$$

while

$$\lim_{h\to\infty} \beta' A_h = 0.$$

Hence, the long-run generalized impulse responses in levels depend on the long-run impact matrix C and converge to finite matrix, while the long-run generalized responses for the cointegration relations converge to zero. The asymptotic distribution of CB is readily determined from the above results and those in, e.g., Paruolo (1997) regarding the asymptotic distribution for the ML estimator of C; see also Johansen (1996).

Specifically, letting $A = CB$ then

$$\sqrt{T}\Big(\mathrm{vec}\big(\hat{A}\big) - \mathrm{vec}\big(A\big)\Big) \xrightarrow{\mathrm{d}} N_{p^2}(0, \Sigma_A),$$

where

$$\Sigma_A = \big[B' \otimes I_p\big] \frac{\partial \mathrm{vec}(C)}{\partial \theta'} \Sigma_\theta \left(\big[B' \otimes I_p\big] \frac{\partial \mathrm{vec}(C)}{\partial \theta'} \right)' + \big[I_p \otimes C\big] \frac{\partial \mathrm{vec}(B)}{\partial \omega'} \Sigma_\omega \left(\big[I_p \otimes C\big] \frac{\partial \mathrm{vec}(B)}{\partial \omega'} \right)'.$$

The matrix with partial derivatives $\partial \mathrm{vec}(B) / \partial \omega'$ is given in equation (11). Furthermore, it is readily shown that

$$\frac{\partial \mathrm{vec}(C)}{\partial \theta'} = \big[\xi' \otimes C\big], \tag{13}$$

where ξ is an $p(k-1) \times p$ matrix given by

$$\xi = \begin{bmatrix} C \\ \vdots \\ C \\ \big(\alpha'\alpha\big)^{-1} \alpha' \big(\Gamma C - I_p\big) \end{bmatrix}.$$

The generalized impulse responses for z provides us with a tool to measure how quickly the long-run relations converge to their steady state values. Since the p shocks may result in $\beta' A_h e_j \approx 0$ for different h, we may, for example, choose a convergence horizon h^* based on the slowest response.

The generalized impulse responses are equal to impulse responses from a structural VAR when the structural shocks are identified from a recursive structure and Ω is diagonal. In all other circumstances will the generalized impulse responses differ from the impulse responses of a structural VAR.

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